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Orthogonality and Disjointness in Spaces of Measures

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The disjointness of measures and their Hahn-Jordan decomposition are instances of the general notion of minimal decomposition in base normed spaces. The mixing distance, a specification of a novel concept of angle in real normed vector spaces, is applied to provide a geometric interpretation of disjointness as orthogonality.

Key words: *Ordered vector space, base norm, minimal decomposition, measure cone, direction distance, mixing distance.*

0. Introduction

The convex and metric structures underlying probabilistic physical theories are generally described in terms of base normed vector spaces. According to a recent proposal, the purely geometrical features of these spaces are appropriately represented in terms of the notion of *measure cone* and the *mixing distance* [1], a specification of the novel concept of *direction distance* [2]. It turns out that the base norm is one member of a whole characteristic family of *mc-norms* from which it can be singled out by virtue of a certain orthogonality relation. The latter is seen to be closely related to the concept of *minimal decomposition*. These connections suggest a simple geometric interpretation of the familiar notion of the disjointness of (probability) measures and the Hahn-Jordan decomposition of measures which has been addressed briefly in [1] and will be elaborated here. The results obtained give an indication of the extent to which a general

measure cone admits measure theoretic interpretations. For the sake of a self-contained presentation, the basic structures are briefly reviewed first.

1. Direction Distance and Mixing Distance in Measure Cones

1.1. In a fundamental investigation [2] Ernst Ruch proposed an interesting extension of Felix Klein's geometry programme for affine geometries based on normed real vector spaces V. Defining the figure of an *(oriented) angle* as an ordered pair $\big([x],[y]\big)$ of directions (rays, $[x] := \{\lambda x | \lambda > 0\}$ for $x \neq 0$), two angles are called *norm-equivalent*, $\big([x],[y]\big) \sim \big([x'],[y']\big)$, if there exists a linear map that sends $[x] \cup [y]$ onto $[x'] \cup [y']$ such that corresponding pairs of points are equidistant: $\| \alpha x_0 - \beta y_0 \| = \| \alpha x_0' - \beta y_0' \|$ for all $\alpha, \beta \in \mathbb{R}^+$. (Here $x_0 := x / \| x \|$, etc.).

The *direction distance* $d[x/y]$ (from x to y) is defined as the family of distances $||\alpha x_0 - \beta y_0||$. This is succinctly summarized in terms of the map

$$d : \big([x],[y]\big) \mapsto d[x/y], \quad d[x/y] : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+, \quad (\alpha, \beta) \mapsto \|\alpha x_0 - \beta y_0\|, \tag{1}$$

The direction distance induces an ordering of the classes of norm-equivalent angles via

$$d[x/y] \succ d[x'/y'] \;:\Longleftrightarrow\; \forall \alpha, \beta \in \mathbb{R}^+ : \|\alpha x_0 - \beta y_0\| \geq \|\alpha x_0' - \beta y_0'\|. \tag{2}$$

1.2. Definition. Let V be a normed real vector space. Elements $z, z' \in V$ are called (*norm-*)*orthogonal*, $z \perp z'$, if the following condition is satisfied:

$$\|\alpha z_0 - \beta z_0'\| \quad = \quad \|\alpha z_0 + \beta z_0'\| \qquad \forall \alpha, \beta \in \mathbb{R}. \tag{3}$$

This notion of orthogonality is based on the equivalence of an angle $([z],[z'])$ and its (planar) complement $([z],[-z'])$: $d[z/z'] = d[z/-z']$.

1.3. With the above definitions the concept of angle is a straightforward generalization of the canonical notion of angle in inner product (pre-Hilbert) spaces. The inner product metric of angles is characterized by the requirement that the direction distance is a symmetric function of its arguments, or equivalently, that the ordering of angles (2) is a total ordering, or equivalently, that the group of linear isometries acts transitively on the norm unit sphere [2]. In an inner product space the orthogonality condition (3) is equivalent to $||z_0 - z_0'|| = ||z_0 + z_0'||$ and to the vanishing of the inner product

between z_0 and z_0'. In the present context another family of normed vector spaces will be considered which are characterized by the fact that only a reduced symmetry is present: a transitive action of a group of isometries is given only on a certain subset of the norm unit sphere.

1.4. Definition. A triple (V, V^+, e) is a measure cone if the following postulates are satisfied:

(a) V is a real vector space with convex, generating cone V^+ $(V = V^+ - V^+)$.

(b) $e : V \to \mathbb{R}$ is a linear functional, called *charge*, that is strictly positive,

$$z \in V^+ \implies \big\{ e(z) \geq 0 \text{ and } e(z) = 0 \Leftrightarrow z = 0 \big\}. \tag{4}$$

It follows that the charge e admits a decomposition $e = e_+ - e_-$ of e into a difference of nonlinear, positive functionals $e_\pm$, where

$$\begin{aligned} e_+ : V &\to \mathbb{R}^+, \quad z \mapsto e_+(z) := \inf\big\{ e(x) |\ x \in V^+,\ x - z \in V^+ \big\}, \\ e_- : V &\to \mathbb{R}^+, \quad z \mapsto e_-(z) := \inf\big\{ e(y) |\ y \in V^+,\ z + y \in V^+ \big\}. \end{aligned} \tag{5}$$

Further, it is required that e marks the cone contour:

$$z \in V^+ \iff e(z) = e_+(z). \tag{6}$$

A measure cone $\big(V, V^+, e\big)$ is said to be a measure cone with *minimal decomposition* if in addition the following postulate is satisfied:

(c) To any $z \in V$ there exists a decomposition $z = z_+ - z_-$, $z_+,\ z_- \in V^+$ such that the following holds: $e(z_+) = e_+(z)$, $e(z_-) = e_-(z)$. Any decomposition of z with this property is called a minimal decomposition of z.

A real vector space V equipped with a measure cone $\big(V, V^+, e\big)$ (with minimal decomposition) will be called mc-space (with minimal decomposition).

All known physically relevant examples of measure cones are equipped with a minimal decomposition. Hence in the sequel the term measure cone will be taken to always include the existence of a minimal decomposition.

1.5. The structure of a measure cone and its properties will be briefly summarized; details and proofs can be found in [1]. The set V^+ is a proper (convex) cone so that

V becomes an ordered vector space via $z \geq z' \quad :\Leftrightarrow \quad z - z' \in V^+$. The strict positivity of the charge functional e ensures that the intersection K of the hyperplane $\{z \in V | e(z) = 1\}$ with V^+ is a base of the convex cone V^+. In a measure cone with minimal decomposition the cone contour condition (6) is a consequence of the strict positivity of e.

Any vector space V associated with a measure cone can be equipped with a norm. In general a triple $\left(V, V^+, e\right)$ consisting of a real vector space V, a convex generating cone $V^+ \subset V$ and a linear functional e is a measure cone if and only if there exists a norm $\|\cdot\|$ marking the cone contour in the following sense:

$$z \in V^+ \quad \Longleftrightarrow \quad e(z) = \|z\|. \tag{7}$$

In particular, the following is a norm of this type, called 1-norm:

$$\|z\|_1 \quad := \quad e_+(z) + e_-(z), \tag{8}$$

This norm turns out to coincide with the base norm, defined as the Minkowski functional of the (absorbing, balanced, radially bounded) set $B := \mathrm{co}(K \cup -K)$, the convex hull of $K \cup -K$; thus, $\|z\|_1 \;=\; \inf\{\lambda \geq 0 | z \in \lambda B\}$ (cf. [3]).

Measure theoretic considerations suggest (cf. [1]) to define a family of norms of the form

$$\|z\| \;:=\; \mathcal{P}\big(e_+(z), e_-(z)\big) \tag{9}$$

which in addition are required to mark the cone contour in the sense of Eq. (7). Such norms, called *mc-norms*, have been completely characterized [1]. A mapping of the form (9) is a norm if and only if the function $\mathcal{P}$, defined on $\mathbb{R}^+ \times \mathbb{R}^+$, has an extension to a norm on $\mathbb{R}^2$ and possesses the symmetry $\mathcal{P}(a,b) = \mathcal{P}(b,a)$. Any such norm is an mc-norm if and only if $\mathcal{P}$ is strictly monotonic in the following sense:

$$\delta > 0 \implies \mathcal{P}(a+\delta, b+\delta) > \mathcal{P}(a,b) \qquad \text{for all } (a,b) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

An mc-space equipped with an mc-norm is called an mc-normed space. The 1-norm is characterized as the mc-norm with the smallest norm unit ball:

$$|e(z)| \quad \leq \quad \|z\| \;=\; \mathcal{P}\big(e_+(z), e_-(z)\big) \quad \leq \quad \|z\|_1.$$

1.6. The *mixing distance* is defined as the restriction of the direction distance to pairs of directions in the positive cone V^+ of an mc-normed space V. All mc-norms associated with a given measure cone are equivalent in that they characterize the same ordering of angles between pairs of directions in V^+ (Theorem 3.1 of [1]). Nevertheless the 1-norm is singled out by the notion of orthogonality [Proposition 2.2 below].

1.7. The use of a measure theoretic terminology in the context of mc-spaces is motivated by the following considerations. A base normed space $\big(V, \|\cdot\|_1\big)$ and its dual order unit space (V', e) provide a general probabilistic framework that was found useful for physical applications and is referred to as a *statistical duality* (cf. [4] and references therein). The set $K \subset V$ is taken to be the convex set of statistical states of a physical system. The positive part $E := [o, e]$ of the order unit interval of V' is a partially ordered, convex set of positive linear functionals on V. E is called the set of *effects* as it represents the totality of yes-no propositions that can be made about a given physical system. E is equipped with a complement operation, $a \mapsto a' := e - a$. Now the elements z of V^+ (or of K), considered as linear functionals on V' via $z(a) := a(z)$, act as positive, additive $[z(a + b) = z(a) + z(b)$ whenever $a + b \in E]$ (and normalized, $z(e) = 1$, if $z \in K$) functions on E, representing thus (probability) measures in a generalized sense.

1.8. Definition. Elements $a, b \in E$ are *(weakly) orthogonal*, $a \perp b$, if their sum is in E again, that is, if $b \le a'$.

The map $a \mapsto a'$ is not an orthocomplementation: one has $(a')' = a$ and $a \le b \Rightarrow b' \le a'$ but not in general $a \wedge a' = o$. [For example, if $a = \frac{1}{2}e$ then $a' = a$]. Here $a \wedge b$ denotes the greatest lower bound of a, b. Effects a, b are called *disjoint* if $a \wedge b = o$. Hence weak orthogonality of effects does not in general imply their disjointness.

2. Orthogonality in a Measure Cone

2.1. The ordered set of classes of equivalent angles induced by the direction distance has a smallest and a largest element represented by the pairs $([x], [x])$ and $([x], [-x]))$, respectively. Orthogonality for a pair of directions in the cone V^+ is equivalent to maximality of the corresponding angle: for $x, y \in K$, $x \perp y$ iff $d[x/y](\alpha, \beta) = \alpha + \beta = d[x/-x]$ (for all α, β). The following Proposition gives a geometric interpretation of the minimal decomposition in relation to the 1-norm [1].

2.2. Proposition. Let (V, V^+, e) be a measure cone.

(1) A decomposition $z = z_+ - z_-$ of $z \in V$ with $z_+, z_- \in V^+$ is a minimal decomposition iff it is an orthogonal decomposition with respect to the 1-norm [that is, $z_+ \perp_1 z_-$] iff $\|z_+ - z_-\|_1 = \|z_+ + z_-\|_1$.

(2) Let $\|\cdot\|_{\mathcal{P}}$ be a norm of the form (9) such that $x \perp y$ for at least one pair $x, y \in K$. Then $\|\cdot\|_{\mathcal{P}} = \|\cdot\|_1$. Thus the 1-norm is the only mc-norm allowing for a nonempty orthogonality relation on $V^+ \times V^+$.

2.3. Definition. Let V be an mc-space equipped with the 1-norm, $E = [o, e] \subset V'$ the associated set of effects. Elements $x, y \in V^+$ are called *disjoint*, $x \dot{\perp} y$, if there exists $a_x \in E$ and $a_y \in E$ such that $a_x \perp a_y$ and $a_x(x) = e(x)$, $a_y(y) = e(y)$ (and hence $a_x(y) = 0 = a_y(x)$).

Note that a_y can be chosen to be $a_y = a_x'$. In fact $x \dot{\perp} y$ iff there exists $a_x \in E$ such that $a_x(x) = e(x)$, $a_x(y) = 0$. This definition captures the idea that disjoint pairs of (positive) measures are carried by weakly orthogonal effects.

2.4. Theorem. Let V be an mc-space equipped with the 1-norm. For any pair $x, y \in V^+$, $x \perp y$ if and only if $x \dot{\perp} y$.

Proof. The following facts will be employed. First, the norm unit ball of the order unit space V' coincides with the order unit interval $[-e, e]$ [3]. Next, as a consequene of the Hahn-Banach theorem, for $z \in V$ there exists $a_z \in [-e, e]$ such that $a_z(z) = \|z\|_1$.

Let $x \perp y$ for some $x, y \in V^+$. By Proposition 2.2, $z = x - y$ is a minimal decomposition of z. Hence, $a_z(z) = a_z(x) - a_z(y) = \|z\|_1 = e(x) + e(y)$ for some a_z, $-e \leq a_z \leq e$. It follows that $a_z(x) = e(x)$ and $a_z(y) = -e(y)$. Define $a_x \in E = [o, e]$ as $a_x := (e + a_z)/2$, then $a_x(x) = e(x)$, $a_x(y) = 0$ and therefore $x \dot{\perp} y$.

Conversely, let $x \dot{\perp} y$ for some $x, y \in V^+$. Taking a_x as in Definition 2.3, define $a := 2a_x - e \in [-e, e]$. It follows that $a(x-y) = 2a_x(x-y) - e(x-y) = e(x) + e(y) = \|x+y\|_1$. On the other hand, one generally has $a(x-y) \leq \|x-y\|_1 \leq \|x+y\|_1$. Therefore, $\|x-y\|_1 = \|x+y\|_1 = e(x) + e(y)$, which is to say that $z = x - y$ is a minimal decomposition and hence $x \perp y$. This completes the proof.

This theorem generalizes the measure theoretic fact that the disjointness of positive measures can be characterized as an orthogonality relation with respect to the total

variation norm. Similarly it comprises the quantum mechanical case where two density operators are disjoint if the projections onto their ranges are mutually orthogonal, and this property is equivalent to the orthogonality of these density operators with respect to the trace norm.

2.5. In the context of measure theory and in the case of quantum mechanical density operators the minimal decomposition into a difference of orthogonal positive components is known to be unique (the Hahn-Jordan decomposition or the spectral decomposition according to the positive and negative parts of the spectrum). However there exist measure cones which do admit non-unique minimal decompositions. For example [1], consider the measure cone generated by $V = \mathbb{R}^3$ and K the square in the plane $z = 1$ with vertices $(\pm 1, \pm 1, 1)$. Then the vector $(1, 0, 0)$ has the following family of minimal decompositions: $(1, 0, 0) = \frac{1}{2}(1, \alpha, 1) - \frac{1}{2}(-1, \alpha, 1)$, $-1 \leq \alpha \leq 1$. This is verified by observing that the vectors $(1, \alpha, 1)$, $(-1, \alpha, 1) \in K$ are orthogonal with respect to the 1-norm and then applying Proposition 2.2. A geometric uniqueness criterion for minimal decompositions in terms of an intersection property has been given in [5]. Uniqueness also holds in an mc-space V which is a vector lattice under the ordering induced by its positive cone V^+ (classical case), since in that case the unique lattice theoretical orthogonal decomposition coincides with the minimal decomposition [6]. Here a measure theoretic condition will be derived which covers classical and quantum probabilistic theories as well as intermediate cases. In general this uniqueness can be ascertained for any mc-space with the property that to every vector there corresponds a kind of support functional in the set of effects.

2.6. Definition. An mc-space V (equipped with the 1-norm) is said to have a *support family* if its dual space V' contains a family of *support functionals* s_z, $z \in V$, characterized by the following conditions:

$$s_z \in [o, e], \tag{S1}$$

$$\|z\|_1 = s_z(z_+ + z_-), \tag{S2}$$

$$a \in [o, e],\ a(z_+ + z_-) = \|z\|_1 \ \Rightarrow\ s_z \leq a, \tag{S3}$$

$$x_1, x_2 \in V^+ \ \Rightarrow\ s_{x_1 - x_2} \leq s_{x_1 + x_2}, \tag{S4}$$

$$x_1, x_2 \in V^+,\ s_{x_1} \perp s_{x_2} \ \Rightarrow\ s_{x_1} \wedge s_{x_2} = o. \tag{S5}$$

Here $z = z_+ - z_-$ denotes a minimal decomposition of z.

2.7. It will be shown below that in an mc-space V with support family there exists only one minimal decomposition for each $z \in V$. At the present stage it is appropriate to observe that the defining conditions for the support functionals are independent of the particular choice of a minimal decomposition (if there were a choice): Let $z = z_+ - z_- = z'_+ - z'_-$ denote two minimal decompositions of z. Then $s_z(z_+) + s_z(z_-) = e(z_+) + e(z_-)$, so that in view of $s_z(z_\pm) \le e(z_\pm)$ it follows that $s_z(z_\pm) = e(z_\pm)$. Similarly, $s_z(z'_\pm) = e(z'_\pm)$. But $e(z_\pm) = e(z'_\pm)$, hence $s_z(z_\pm) = s_z(z'_\pm)$.

Property $(S1)$ is motivated by the fact that the support of a measure is represented by a characteristic function and hence an extreme effect; similarly, the support of a self-adjoint (trace class) operator is given by a projection operator, which is an extreme effect. $(S2)$, $(S3)$ and $(S4)$ capture the idea that a support is a minimal carrier. In particular it follows from $(S2)$, $(S3)$ that for any $z \in V$ the support $s_z \in E$ is unique; so there exists only one map $z \mapsto s_z$. $(S5)$ is a strengthening of the weak orthogonality of two support effects into disjointness.

2.8. Proposition. Let V be an mc-space with support family. Then for $z \in V$, $x_1, x_2 \in V^+$ the following statements hold:

(i) $$z = 0 \iff s_z = o;$$

(ii) $$x_1 \le x_2 \implies s_{x_1} \le s_{x_2};$$

(iii) $$s_{x_1} \le s_{x_2} \iff s_{x_2} = s_{x_1+x_2};$$

(iv) $$x_1 \dot{\perp} x_2 \iff s_{x_1} \perp s_{x_2} \implies s_{x_1+x_2} \le s_{x_1} + s_{x_2}.$$

Proof. Ad (i): If $s_z = o$ then by $(S2)$: $z = 0$. If $z = 0$ then $z_+ = z_- = 0$ and so for $a = o$: $a(z_+ + z_-) = ||z||_1 = 0$, hence by $(S3)$ $s_z \le a$ and so by $(S1)$, $s_z = o$.

Ad (ii): Assume $x_1 \le x_2$. Then for $a \in [o, e]$, $a(x_2) = 0$ implies $a(x_1) = 0$. For $a = e - s_{x_2}$ one has $a(x_2) = 0$, hence $s_{x_2}(x) = e(x)$. By $(S3)$, $s_{x_1} \le s_{x_2}$.

Ad (iii): The relations $s_{x_1} \le sx_1 + x_2$, $s_{x_2} \le s_{x_1+x_2}$ follow from (ii). Let $s_{x_1} \le s_{x_2}$, then $s_{x_2}(x_1 + x_2) = \|x_1 + x_2\|_1$, so by $(S3)$, $s_{x_1+x_2} \le s_{x_2}$ and therefore $s_{x_2} = s_{x_1+x_2}$. The converse implication is obvious in view of $s_{x_1} \le s_{x_1+x_2}$.

Ad (iv): Let $x_1 \dot{\perp} x_2$. Thus there is $a_{x_1} \in [o, e]$ such that $a_{x_1}(x_1) = e(x_1)$ and $a_{x_1}(x_2) =$

0. Then for $a_{x_2} = e - a_{x_1}$ one has $a_{x_2}(x_2) = e(x_2)$, $a_{x_2}(x_1) = 0$. So by $(S3)$, $s_{x_1} \leq a_{x_1}$, $s_{x_2} \leq a_{x_2}$. Therefore $s_{x_1} + s_{x_2} \leq a_{x_1} + a_{x_2} = e$, that is, $s_{x_1} \perp s_{x_2}$.
Let $s_{x_1} \perp s_{x_2}$. Then $s_{x_1}(x_1) = e(x_1)$, so $s_{x_2}(x_1) = 0$, and also $s_{x_2}(x_2) = e(x_2)$, so $s_{x_1}(x_2) = 0$. This is to say that $x_1 \dot{\perp} x_2$. It also follows that $s_{x_1}+s_{x_2}(x_1+x_2) = e(x_1+x_2)$, so by $(S3)$, $s_{x_1+x_2} \leq s_{x_1} + s_{x_2}$. This completes the proof.

2.9. Remark. It is straightforward to see that $s_x = s_{\lambda x}$ for $x \in K$, $\lambda \neq 0$, and $s_z = s_{z_+ + z_-}$ for $z \in V$. Hence the set of supports s_x of elements $x \in K$ exhaust the whole family of supports. For any $x \in K$, define $F_x := \{y \in K | s_y \leq s_x\}$. These sets are faces of the convex set K, and the map $s_x \mapsto F_x$ is an order isomorphism where the set of faces F_x is ordered by set inclusion. Furthermore, for any two supports s_x, s_y the supremum exists and is $s_x \vee s_y = s_{x+y} = s_{\frac{1}{2}x+\frac{1}{2}y}$. So $F_x \vee F_y = F_{\frac{1}{2}x+\frac{1}{2}y}$. In the classical and quantum cases, the support family is given by the extreme effects, that is, a Boolean algebra of sets underlying the classical theory, and the orthocomplemented lattice of projections (closed subspaces) of the underlying Hilbert space, respectively; and these lattices are isomorphic to the corresponding lattices of (closed) faces of the respective sets K of probability measures or density operators.

2.10. Theorem. Let V be an mc-space (equipped with the 1-norm) with support family. Then the minimal decomposition $z = z_+ - z_-$ of any element $z \in V$ is uniquely determined.

Proof. Let $z = z_+ - z_- = z'_+ - z'_-$. It follows that $z_+ - z'_+ = z_- - z'_-$. By $s_{z_\pm} = s_{z'_\pm}$ and Proposition 2.8. (iii) one has $s_{z_\pm + z'_\pm} = s_{z_\pm} = s_{z'_\pm}$. Further, by $(S4)$, $s_{z_\pm - z'_\pm} \leq s_{z_\pm + z'_\pm}$. Therefore, $s_{z_\pm - z'_\pm} \leq s_{z_+}$ and $\leq s_{z_-}$. Now $s_{z_+} \perp s_{z_-}$ [by Proposition 2.8. (iv)], so that finally $(S5)$ yields $s_{z_+ - z'_+} = s_{z_- - z'_-} = o$, that is, by Proposition 2.8. (i), $z_+ = z'_+$, $z_- = z'_-$. This completes the proof.

3. Orthogonality-preserving mc-endomorphisms.

3.1. The mappings on an mc-space V which respect the structure of the measure cone, the mc-endomorphisms, are the linear, charge-preserving, positive mappings on V. These mappings are contractions with respect to any mc-norm. In particular, they lead to decreasing mixing distance. Conversely, any linear, charge preserving contraction is necessarily a positive mapping [1]. With the above results it is straightforward to verify

the following statement.

3.2. Proposition. Let V be an mc-space equipped with the 1-norm. The orthogonality-preserving mc-endomorphisms on V are exactly the positive isometries on V.

Proof. Let Φ be an orthogonality-preserving mc-endomorphism. Then

$$\begin{aligned} \|\Phi z\|_1 &= e(\Phi z_+) + e(\Phi z_-) \qquad [\Phi \text{ positive, orthogonality-preserving}] \\ &= e(z_+) + e(z_-) \qquad [\Phi \text{ charge-preserving}] \\ &= \|z\|_1 \end{aligned}$$

Conversely, assume Φ to be a positive isometry, then for a minimal decomposition of z, $z = z_+ - z_-$ one has

$$e(\Phi z_+) + e(\Phi z_-) = e(z_+) + e(z_-) = ||z_+ - z_-||_1 = ||\Phi z_+ - \Phi z_-||,$$

so that $\Phi z = \Phi(z_+) - \Phi(z_-)$ is a minimal decomposition as well. By Proposition 2.2 this is to say that any orthogonal pair $x, y \in V^+$ is mapped onto an orthogonal pair. This completes the proof.

3.3. The last result can be elaborated into an operational characterization of the (ir)reversibility of a statistical state transformation if the set K is interpreted as a set of statistical states of a physical system and the mc-endomorphisms are interpreted as stochastic operators on V [7]. A particularly interesting problem is to specify those mc-spaces for which the following implication is always true: for states $x, y, x', y, \in K$, $d[x/y] \succ d[x'/y']$ implies that there exists a linear state transformation Φ such that $(x', y') = (\Phi x, \Phi y)$. Partial answers are reviewed in [1] and [2], and the general case in the context of classical probability theory is demonstrated in [7]. It is also known that the above implication does not hold in general. Still a particular special case pertains to arbitrary mc-spaces.

3.3. Proposition. Let V be an mc-space equipped with the 1-norm. Let $x, y \in K$, $x \perp y$. Then $d[x/y] \succ d[x'/y']$. Take a_x, a_y (as in Definition 2.3) so that $a_x + a_y = e$, $a_x(x) = a_y(y) = e(x) = e(y) = 1$. Then $\Phi : z \mapsto a_x(z)x' + a_y(z)y'$ is an mc-endomorphism, and $\Phi x = x'$, $\Phi y = y'$.

4. Conclusion.

The structure of a measure cone (equivalently, a base normed space) with minimal decomposition allows for a natural geometric interpretation of the disjointness of measures defined in terms of the minimal decomposition. Components of a minimal decomposition are mutually orthogonal (Proposition 2.2). Furthermore, orthogonal measures are carried by weakly orthogonal effects (Theorem 2.4). In a measure cone with support family the minimal decompositions are unique (Theorem 2.10). Finally the notion of orthogonality yields a characterization of *isometric* mc-endomorphisms, i.e., reversible state transformations in the sense of [8]. In the case of quantum mechanical systems it has proved very useful in deriving the general form of isometric state transformations [9]. In this way the concept of orthogonality developed here is found to provide some insight into the interpretation of (ir)reversibility in the sense of decreasing mixing distance.

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